The Leading Coefficient Test

In Example 1, note that the three graphs eventually rise or fall without bound as x moves to the right or left. Symbolically, we write

$$f(x) \to \infty \quad \text{as} \quad x \to \infty$$

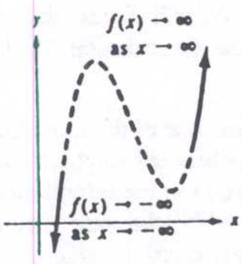
to mean that $f(x)$ increases without bound as x moves to the right without bound. (The infinity symbol ∞ indicates unboundedness.) Whether the graph of a polynomial eventually rises or falls can be determined by the function's degree (even or odd) and by its leading coefficient, as indicated by the **Leading Coefficient Test.**

A review of the shapes of the graphs of polynomial functions of degree 0, 1, and 2 may be used to illustrate the Leading Coefficient Test.

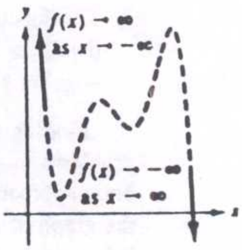
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function $f(x) = a_n x^n + \cdots + a_1 x + a_0$ eventually rises or falls in the following manner. (*Note:* The dashed portions of the graphs indicate that the test determines *only* the right and left behavior of the graph.)

1. **When n is *odd*:**

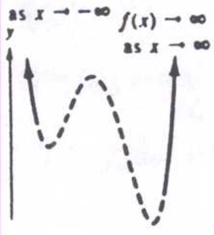


If the leading coefficient is positive ($a_n > 0$), then the graph falls to the left and rises to the right.

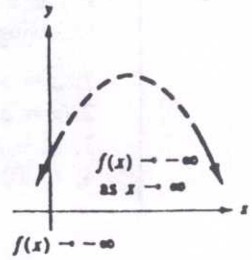


If the leading coefficient is negative ($a_n < 0$), then the graph rises to the left and falls to the right.

2. **When n is *even*:**



If the leading coefficient is positive ($a_n > 0$), then the graph rises to the left and right.



If the leading coefficient is negative ($a_n < 0$), then the graph falls to the left and right.

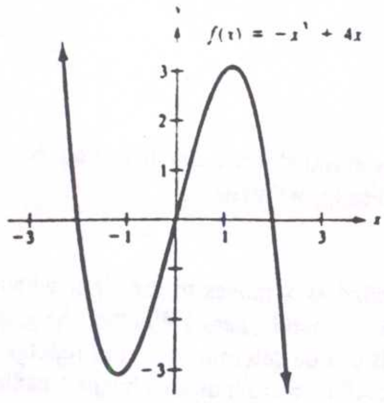


FIGURE 4.19

EXAMPLE 2 Applying the Leading Coefficient Test

Use the Leading Coefficient Test to determine the right and left behavior of the graphs of the polynomial functions.

a. $f(x) = -x^3 + 4x$ **b.** $f(x) = x^4 - 5x^2 + 4$ **c.** $f(x) = x^5 - x$

Solution

a. Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 4.19.

b. Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in Figure 4.20.

c. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in Figure 4.21.

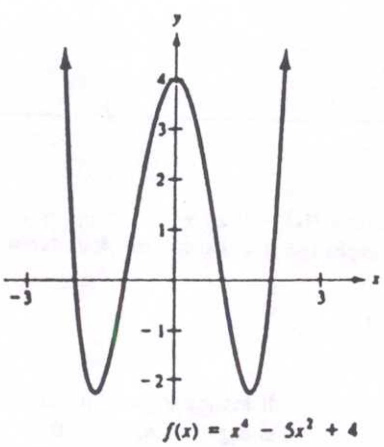


FIGURE 4.20

Zeros of Polynomial Functions

A zero of a function f is a number x for which $f(x) = 0$. For instance, 2 is a zero for the function $f(x) = x - 2$ because $f(2) = 2 - 2 = 0$. Similarly, 0 and -3 are zeros of the function $f(x) = x^2 + 3x$ because $f(0) = 0^2 + 3(0) = 0$ and $f(-3) = (-3)^3 + 3(-3) = 0$.

It can be shown that for a polynomial function f of degree n, the following statements are true.

1. The graph of f has, at most, $n - 1$ turning points. (Turning points are points at which the graph changes from increasing to decreasing or vice versa.)
2. The function f has, at most, n real zeros. (We will discuss this result in detail in Section 4.5 when we present the Fundamental Theorem of Algebra.)

Finding the zeros of polynomial functions is one of the most important problems in algebra. There is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros, and in other cases you can use information about the zeros of a function to help sketch its graph.

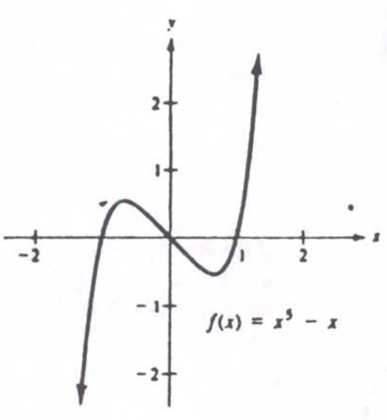


FIGURE 4.21

REAL ZEROS OF POLYNOMIAL FUNCTIONS

If f is a polynomial function and a is a real zero of f, then the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.
2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
4. $(a, 0)$ is an *x-intercept* of the graph of f.

Finding zeros of polynomial functions is closely related to factoring and finding x-intercepts, as demonstrated in Examples 3, 4, and 5.

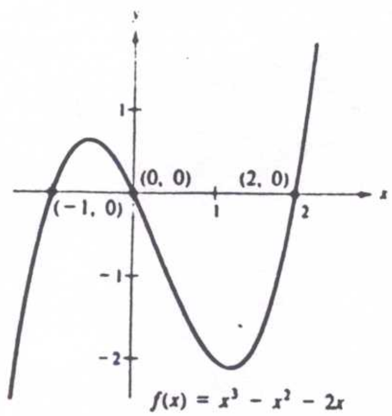


FIGURE 4.22

EXAMPLE 3 Finding Zeros of a Third-Degree Polynomial Function

Find all real zeros of $f(x) = x^3 - x^2 - 2x$.

Solution

$$f(x) = x^3 - x^2 - 2x \quad \textit{Given function}$$
$$= x(x^2 - x - 2) \quad \textit{Remove common monomial factor}$$
$$= x(x - 2)(x + 1) \quad \textit{Factor completely}$$

Thus, the real zeros are $x = 0$, $x = 2$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(2, 0)$, and $(-1, 0)$, as shown in Figure 4.22. In the figure, note that the graph has two turning points. This is consistent with the fact that a third-degree polynomial can have *at most* two turning points.

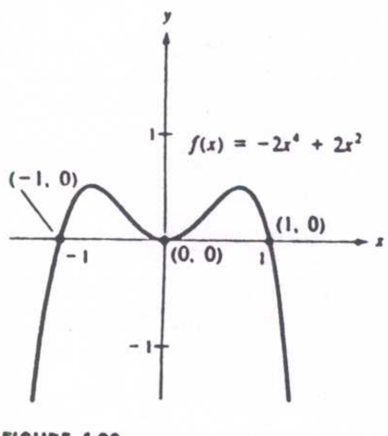


FIGURE 4.23

EXAMPLE 4 Finding Zeros of a Fourth-Degree Polynomial Function

Find all real zeros of $f(x) = -2x^4 + 2x^2$.

Solution

$$f(x) = -2x^4 + 2x^2 \quad \textit{Given function}$$
$$= -2x^2(x^2 - 1) \quad \textit{Remove common monomial factor}$$
$$= -2x^2(x - 1)(x + 1) \quad \textit{Factor completely}$$

Thus, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(1, 0)$, and $(-1, 0)$, as shown in Figure 4.23. Note in the figure that the graph has three turning points, which is consistent with the fact that a fourth-degree polynomial can have at most three turning points.

If you have a graphing utility available, consider approximating the real zeros of a function such as $f(x) = x^5 - 4x^3 + 4x$.

In Example 4, the real zero arising from $-2x^2 = 0$ is a **repeated zero.** In general, we say that a factor $(x - a)^k$ yields a repeated zero $x = a$ of **multiplicity** k. If k is odd, then the graph *crosses* the x-axis at $x = a$. If k is even, then the graph *touches* (but does not cross) the x-axis at $x = a$, as shown in Figure 4.23.