

Matrices and Systems of Linear Equations

Topics: Elementary Row Operations • Gaussian Elimination with
Substitution • Gauss-Jordan Elimination

Matrices

In this section you will study a streamlined technique for solving systems of linear equations. This technique involves the use of a rectangular array of real numbers called a matrix.

Definition of a Matrix

If m and n are positive integers, an $m \times n$ matrix (read “ m by n ”) is a rectangular array

$$\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array} \right] \quad \begin{array}{l} m \text{ rows} \\ n \text{ columns} \end{array}$$

in which each entry, a_{ij} , of the matrix is a number. An $m \times n$ matrix has m rows (horizontal lines) and n columns (vertical lines).

The entry in the i th row and j th column is denoted by the *double subscript* notation a_{ij} . A matrix having m rows is said to be of **order** $m \times n$. If $m = n$, the matrix is **square** of order n . For a square matrix, the entries a_{11} , a_{22} , a_{33} , \dots are the **main diagonal** entries.

EXAMPLE 1 Examples of Matrices

The following matrices have the indicated orders.

a. Order: 1×1

$$[2]$$

b. Order: 1×4

$$\left[1 \quad -3 \quad 0 \quad \frac{1}{2} \right]$$

c. Order: 2×2

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

d. Order: 3×2

$$\begin{bmatrix} 5 & 0 \\ 2 & -2 \\ -7 & 4 \end{bmatrix}$$

NOTE The plural of matrix is *matrices*.

NOTE A matrix that has only one row is called a **row matrix**, and a matrix that has only one column is called a **column matrix**.

The last matrix in Example 3 is said to be in row-echelon form. The term *echelon* refers to the stair-step pattern formed by the nonzero elements of the matrix. To be in this form, a matrix must have the following properties.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in row-echelon form has the following properties.

1. All rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a leading 1).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in *row-echelon form* is in *reduced row-echelon form* if every column that has a leading 1 has zeros in every position above and below its leading 1.

EXAMPLE 4 Row-Echelon Form

The following matrices are in row-echelon form.

a.
$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c.
$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

d.
$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrices in (b) and (d) also happen to be in *reduced* row-echelon form. The following matrices are not in row-echelon form.

e.
$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

f.
$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

Every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 4, you can change the matrix in part (e) to row-echelon form by multiplying its second row by $\frac{1}{2}$. What elementary row operation could you perform on the matrix in part (f) so that it would be in row-echelon form?

Solution of Linear Systems using Gauss-Jordan Elimination

We use elementary row operations to solve systems of linear equations. These elementary operations are:

1. interchange any two rows
2. Multiply any row by a nonzero constant
3. Add a multiple of a row to another row.

To solve a system of linear equations, we use elementary row operations on the augmented matrix to produce a matrix in fully reduced row echelon form. (located on -
REVERSE SIDE)

Example 1. Solve

$$\begin{array}{rcl} x - 2y + 3z & = & 9 \\ 2x - 5y + 5z & = & 17 \\ -x + 3y & = & -4 \end{array}$$

Example 2. Solve

$$\begin{array}{rcl} x + 2y - z & = & 3 \\ 3x + 7y - 5z & = & 14 \\ -2x - y - 3z & = & 8 \end{array}$$

Example 3. Solve:

$$\begin{array}{rcl} x - y + 2z & = & 4 \\ x & + & z = 6 \\ 2x - 3y + 5z & = & 4 \\ 3x + 2y - z & = & 1 \end{array}$$

Example 4. Solve:

$$\begin{array}{rcl} 2x + 4y - 2z & = & 0 \\ 3x + 5y & = & 1 \end{array}$$

(OVER)

A matrix is in Reduced Echelon Form if all of the following are true:

1. All rows consisting entirely of zeros are grouped at the bottom of the matrix.

2. The left-most nonzero number in each row is 1. This element is called the leading one.

3. The leading 1 of a row is to the right of the leading 1 of the previous row.

4. All entries directly above and below a leading 1 are zeros.

Examples:

a)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

b)
$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

c)
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Facts: Every matrix can be converted to Reduced Echelon Form using a sequence of elementary row operations.

If the augmented matrix representing a linear system of equations is converted to Reduced Echelon Form, then three situations are possible:

1. If a row like $[0 \ 0 \ \dots \ 0|a]$, $a \neq 0$ appears in the matrix, the system has no solution.

2. If the number of nonzero rows equals the number of variables in the system, there is exactly one solution.

3. If the number of nonzero rows is less than the number of variables in the system, there will be infinitely many solutions.