APPLICATION 4.2: Projectiles and Range

See Section 4.3
Increasing and Decreasing Functions and the First Derivative Test
Calculus, 4th Edition,
Larson/Hostetler/Edwards

Suppose you and a friend are competing to see who can throw a stone the farthest. You are standing on level ground. In this classical projectile problem, the maximum range of the stone (distance the stone goes before hitting the ground) can be described in terms of its initial velocity.

Here we consider the more complex case in which the game is played on a hill. Figure 4.2 illustrates the situation: θ is the angle of inclination of the hill, the projectile has an initial velocity v_0, and we assume no air resistance. Two angles, θ and ϕ, affect the distance x the projectile can reach.

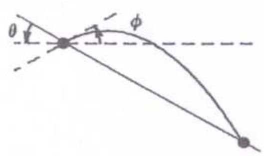

FIGURE 4.2

EXAMPLE

Downhill Toss

For a downhill stone toss, what is the angle ϕ that permits the stone to travel the farthest?

SOLUTION

In order to solve the problem, we must express the impact coordinates in terms of the angle θ. In Figure 4.2 we see that we can decompose the x and y distances by using the angles θ and ϕ. The results are

$$x = v_0(\cos\phi)t$$

and

$$y = v_0(\sin\phi)t - \frac{1}{2}gt^2 = -x\tan\theta.$$

We solve for t in the x-equation.

$$t = \frac{x}{v_0\cos\phi}$$

Substituting this value into the y-equation, we obtain

$$-x\tan\theta = \frac{xv_0\sin\phi}{v_0\cos\phi} - \frac{gx^2}{2(v_0\cos\phi)^2}$$
$$-x\tan\theta - x\tan\phi = -\frac{gx^2}{2(v_0\cos\phi)^2}.$$

The value $x = 0$ represents a minimum distance and is not the solution we seek. Thus, we can divide both sides of the equation by x and solve the remaining equation for x. This leads to

$$x = \frac{2v_0{}^2\cos^2\phi(\tan\phi + \tan\theta)}{g}$$
$$= k(\cos\phi\sin\phi + \cos^2\phi\tan\theta)$$

where $k = 2v_0{}^2/g > 0$. At this point we have an equation expressing x in terms of the two angles θ and ϕ. We seek those values that will maximize x. We differentiate with respect to ϕ. The derivative is

$$\frac{dx}{d\phi} = k(\cos 2\phi - \sin 2\phi\tan\theta)$$

which yields a critical number at

$$\cot 2\phi = \tan\theta.$$

Using a cofunction identity, we obtain

$$\tan\left(\frac{\pi}{2} - 2\phi\right) = \tan\theta$$
$$\frac{\pi}{2} - 2\phi = \theta$$
$$2\phi = \frac{\pi}{2} - \theta$$
$$\phi = \frac{\pi}{4} - \frac{\theta}{2}.$$

We conclude that $\phi = (\pi/4) - (\theta/2)$ for maximum range. Notice that the result reduces to the maximum range result for a horizontal situation when θ approaches zero.

APPLICATION 4.2 EXERCISES

1. Solve a similar problem in which the stone is thrown uphill on a hill with angle of inclination θ.
2. In the given example, assume that the angle of inclination of the hill is $\theta = 20°$. How far downhill can an object whose initial velocity is 64 feet/second be thrown?
3. You are throwing stones off a 100-foot cliff overlooking a lake. At what angle should you throw your stone to achieve the maximum range? Again, assume no air resistance. Also, assume that the initial velocity of the object is 64 feet/second.
4. You and your friends are having a snowball fight. The teams are located on opposite sides of a flat-roofed garage twelve feet tall and twenty feet wide. Suppose the other team throws snowballs at a speed of 64 feet/second. Is there a "safe zone" near the garage in which you can be certain that the other team will not be able to hit you with a snowball? Is so, describe this zone. (Assume the snowballs are thrown from a height of 6 feet.)