APPLICATION 8.6: Pursuit Problem

See Section 8.4
Trigonometric Substitution
Calculus, 4th Edition,
Larson/Hostetler/Edwards

The Dog and Rabbit Problem

A dog spots a rabbit running in a straight line in a field. Fortunately for the rabbit, the distance between the dog and the rabbit remains constant. Assuming that the dog always moves toward the rabbit and that the distance between them is 10 feet, find an equation to describe the path of the dog.

SOLUTION

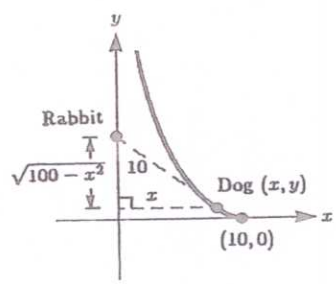

FIGURE 8.4

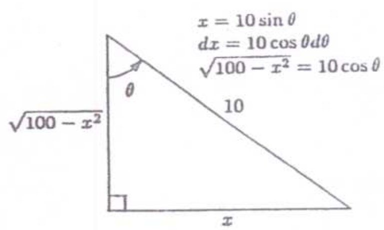

FIGURE 8.5

See Figure 8.4. Let (0, 0) and (10, 0) represent the initial positions of the rabbit and the dog, respectively, and assume that the rabbit moves along the y-axis. Let (x, y) represent the position of the dog. We wish to establish a differential equation involving dy/dx. The slope of the tangent line to the dog's path is

$$\frac{dy}{dx} = -\frac{\sqrt{100-x^2}}{x}.$$

Using trigonometric substitution (as indicated in Figure 8.5), we solve for y as follows.

$$\begin{aligned}
y &= -\int \frac{\sqrt{100-x^2}}{x}\, dx \\
&= -\int \frac{10\cos\theta}{10\sin\theta} 10\cos\theta\, d\theta \\
&= -10\int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta \\
&= -10\int (\csc\theta - \sin\theta)\, d\theta \\
&= -10[-\ln|\csc\theta + \cot\theta| + \cos\theta] + C \\
&= 10\left[\ln\left|\frac{10}{x} + \frac{\sqrt{100-x^2}}{x}\right| - \frac{\sqrt{100-x^2}}{10}\right] + C.
\end{aligned}$$

Next, we use the initial condition $y = 0$ when $x = 10$. So,

$$0 = 10\ln 1 + C$$
$$0 = C.$$

Hence, the equation for the dog's path is

$$\begin{aligned} y &= 10\left[\ln\left|\frac{10}{x} + \frac{\sqrt{100-x^2}}{x}\right| - \frac{\sqrt{100-x^2}}{10}\right] \\ &= 10\ln\left|\frac{10+\sqrt{100-x^2}}{x}\right| - \sqrt{100-x^2}. \end{aligned}$$