APPLICATION 3.5: Melting Snow or Ice

See Section 3.7
Related Rates
Calculus, 4th Edition,
Larson/Hostetler/Edwards

The earth's temperature is rising, causing even the most skeptical observer concern about the effects of global warming if the process continues unchecked. Most observers believe that through cooperative reduction in the amount of carbon dioxide released by human activities; by protection and replanting of land, especially forests; by conservation of fossil fuels; by recycling; and through the application of new technologies to assist these efforts, the "worst-case scenario" can be averted.

Yet, the possible melting of either polar ice cap must be considered. We know that the melting of Antarctic ice is influenced not only by temperature, but also by such factors as existing seasonal variation in the amount of ice, wind activity, water channels, the crystalline structure of ice formed under different conditions, and the closeness of ice to land mass.*

Our problem here is a much simplified melting situation. A spherical snowball exists, formed of pure H_2O, and is melting. We are interested in investigating the volume of the snowball, as a function of time. Suppose it is known that the rate of melting is proportional to the surface area of the snowball, and that the volume can be measured at two different times. We then can determine when the snowball will be completely melted.

EXAMPLE

Snowball

A spherical snowball melts at a rate proportional to its surface area. Its volume decreases from 100 cubic centimeters at noon to 80 cubic centimeters at 1 P. M.

(a) Find the volume as a function of time.

(b) When will the snowball be completely melted?

SOLUTION

Let V represent the volume of the snowball, let S represent the surface area of the snowball, and let r represent the radius of the snowball.

(a) We have $V = \frac{4}{3}\pi r^3$. Then

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

But we also know the snowball melts at a rate proportional to its surface area. Thus,

$$\frac{dV}{dt} = KS$$

and since $S = 4\pi r^2$, we have

$$\frac{dV}{dt} = K(4\pi r^2).$$

Therefore,

$$4\pi r^2 \frac{dr}{dt} = K(4\pi r^2).$$

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*From *Calypso Log*, 1989 (Cousteau Society).

So $dr/dt = K$ is a constant which implies that r is a linear function of t and we can write

$$r = Kt + b.$$

By substitution, we determine K and b. We let $t = 0$ represent the time at noon. Then,

$$100 = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad r = \left(\frac{75}{\pi}\right)^{1/3}$$

which implies that

$$b = \left(\frac{75}{\pi}\right)^{1/3}$$

We let $t = 1$ represent the time at 1 P. M. Then,

$$80 = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad r = \left(\frac{60}{\pi}\right)^{1/3}$$

which implies that

$$\left(\frac{60}{\pi}\right)^{1/3} = K(1) + \left(\frac{75}{\pi}\right)^{1/3}$$

$$K = \left(\frac{60}{\pi}\right)^{1/3} - \left(\frac{75}{\pi}\right)^{1/3}.$$

Finally, it follows that

$$r = \left[\left(\frac{60}{\pi}\right)^{1/3} - \left(\frac{75}{\pi}\right)^{1/3}\right]t + \left(\frac{75}{\pi}\right)^{1/3}$$

and so

$$V = \frac{4}{3}\pi\left(\left[\left(\frac{60}{\pi}\right)^{1/3} - \left(\frac{75}{\pi}\right)^{1/3}\right]t + \left(\frac{75}{\pi}\right)^{1/3}\right)^3.$$

(b) To determine when the snowball will be entirely melted, we can use the fact that, when melting is complete, the snowball's volume, or, equivalently its radius, will be zero.

$$\left[\left(\frac{60}{\pi}\right)^{1/3} - \left(\frac{75}{\pi}\right)^{1/3}\right]t + \left(\frac{75}{\pi}\right)^{1/3} = 0$$

$$t = \frac{(75/\pi)^{1/3}}{(75/\pi)^{1/3} - (60/\pi)^{1/3}} \approx 14 \text{ hours.}$$

Thus, the snowball will be completely melted in about 14 hours.

APPLICATION 3.5 EXERCISES

1. Solve the problem as given in the Example except assume that the volume is 125 cubic centimeters at noon and 64 cubic centimeters at 2 P. M.
2. An ice cube in the shape of a cube melts at a rate proportional to its surface area. If the volume was 125 cubic centimeters at noon and 64 cubic centimeters three hours later, determine the volume of the ice cube as a function of time. When will the ice cube be completely melted?