

1)

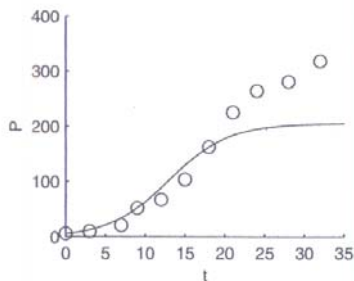
- (a) Use the following three data points $P_0 = P(0) = 6$, $P_1 = P(9) = 52$, and $P_3 = P(18) = 163$ to estimate r and k : Set $h = 9$. Equation (1.17) gives

$$r = \frac{1}{9} \ln \left(\frac{163(52 - 6)}{6(163 - 52)} \right) = 0.269.$$

Then using this in equation (1.16), one obtains

$$K = \frac{(6)(52)(1 - e^{-0.269(9)})}{6 - 52e^{-0.269(9)}} = 205.90.$$

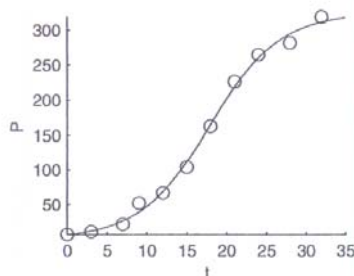
The plot of the logistic solution with these parameters is shown in the following figure.



- (b) To compute the parameters we want to find values of the parameters P_0 , r , and K which minimize

$$\sum_{j=1}^{11} \left[\log \left(\frac{P(t_j)}{P_j} \right) \right]^2,$$

where t_j and P_j are the data from Table 1, and $P(t)$ is the logistic function in equation 3.1.13 with parameters P_0 , r , and K . Using a non-linear least squares minimization program we find that $K = 324.1241$, $P_0 = 5.6047$, and $r = 0.2268$. The plot of the logistic solution with these parameters is shown in the following figure.



- (c) If we use only three points to compute the parameters, the fit of the logistic model is very poor. However, if we use all of the data the fit is quite good.

2)

- (a) Let $P(t)$ represent the balance at the end of t years. Let r represent the annual rate and P_0 the initial investment. Thus,

$$P' = rP, \quad P(0) = P_0.$$

Consequently,

$$\begin{aligned} P(t) &= P_0 e^{rt}, \\ P(10) &= 2000 e^{0.06(10)}, \\ P(10) &\approx \$3,644.24 \end{aligned}$$

- (b) In semiannual case, $m = 2$. Furthermore, there are 20 compounding periods in 10 years, so

$$P(20) = 2000 \left(1 + \frac{0.06}{2} \right)^{20} \approx \$3,612.22.$$

In the monthly case, $m = 12$. There are 120 compounding periods in 10 years, so

$$P(120) = 2000 \left(1 + \frac{0.06}{12} \right)^{120} \approx \$3,638.79.$$

In the daily case, $m = 365$. There are 3650 compounding periods in 10 years, so

$$P(3650) = 2000 \left(1 + \frac{0.06}{365} \right)^{3650} \approx \$3,644.06.$$

3)

Use $k = (mg)/x$ to determine the spring constant.

$$k = \frac{5 \text{ kg} \times 9.8 \text{ m/s}^2}{0.75 \text{ m}} = 65.3 \text{ N/m}.$$

Using the model

$$my'' + \mu y' + ky = F(t),$$

we note that there is no damping ($\mu = 0$) and there is no driving force ($F(t) = 0$), so the equation becomes

$$my'' + ky = 0.$$

With $m = 5 \text{ kg}$ and $k = 65.3 \text{ N/m}$, this becomes

$$5y'' + 65.3y = 0.$$

Because the initial displacement was 36 cm upward, $y(0) = -0.36$ (assuming an orientation where y is positive in the downward direction). Because the mass is given an initial downward velocity of 0.45 m/s, $y'(0) = 0.45$. Because we've chosen the downward direction to represent positive y -values, downward velocities are positive.

4)

Let $y = e^{\lambda t}$ in $y'' + y' - 12y = 0$ to obtain

$$\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 12e^{\lambda t} = 0$$

$$e^{\lambda t}(\lambda^2 + \lambda - 12) = 0.$$

Because $e^{\lambda t} \neq 0$, we arrive at the characteristic equation

$$\lambda^2 + \lambda - 12 = 0$$

$$(\lambda + 4)(\lambda - 3) = 0,$$

and roots $\lambda = -4$ and $\lambda = 3$. Because the roots are distinct, the solutions $y_1(t) = e^{-4t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{-4t} + C_2 e^{3t}.$$

5)

If $y'' + 2y' + 17y = 0$, then the characteristic equation is

$$\lambda^2 + 2\lambda + 17 = 0.$$

The roots of the characteristic equation are $-1 \pm 4i$, leading to the complex solutions

$$z(t) = e^{(-1+4i)t} \quad \text{and} \quad \bar{z}(t) = e^{(-1-4i)t}.$$

However, by Euler's identity,

$$z(t) = e^{-t} e^{4it} = e^{-t}(\cos 4t + i \sin 4t),$$

and the real and imaginary parts of z lead to a fundamental set of real solutions $y_1(t) = e^{-t} \cos 4t$ and $y_2(t) = e^{-t} \sin 4t$. Hence the general solution is

$$y(t) = C_1 e^{-t} \cos 4t + C_2 e^{-t} \sin 4t.$$

6)

If $4y'' + 12y' + 9y = 0$, then the characteristic equation is

$$4\lambda^2 + 12\lambda + 9 = (2\lambda + 3)^2 = 0.$$

Hence, the characteristic equation has a repeated root, $\lambda = -3/2$. Therefore, $y_1(t) = e^{-(3/2)t}$ and $y_2(t) = t e^{-(3/2)t}$ form a fundamental set of real solutions. Hence, the general solution is

$$y(t) = C_1 e^{-(3/2)t} + C_2 t e^{-(3/2)t} = (C_1 + C_2 t) e^{-(3/2)t}.$$

8) concept sheet

7)

If $y'' - 4y' - 5y = 0$, then the characteristic equation is

$$\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0,$$

with roots $\lambda = 5$ and $\lambda = -1$. This leads to the general solution

$$y(t) = C_1 e^{5t} + C_2 e^{-t}.$$

Differentiating the general solution,

$$y'(t) = 5C_1 e^{5t} - C_2 e^{-t},$$

then using $y'(1) = -1$,

$$-1 = 5C_1 e^5 - C_2 e^{-1}.$$

Thus, $C_1 = -(1/3)e^{-5}$ and $C_2 = -(2/3)e$. Thus, the final solution is

$$y(t) = -\frac{1}{3} e^{-5} e^{5t} - \frac{2}{3} e e^{-t} = -\frac{1}{3} e^{5(t-1)} - \frac{2}{3} e^{-(t-1)}.$$

8)

(a) By Hooke's Law,

$$k = \frac{F}{y}$$

$$k = \frac{mg}{y}$$

$$k = \frac{(1 \text{ kg})(9.8 \text{ m/s}^2)}{4.9 \text{ m}}$$

$$k = 2 \text{ N/m}$$

(b) With $m = 1 \text{ kg}$, $\mu = 3 \text{ kg/s}$, and $k = 2 \text{ N/m}$, $my'' + \mu y' + ky = 0$ becomes

$$y'' + 3y' + 2y = 0,$$

with characteristic equation $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$ and zeros $\lambda_1 = -1$ and $\lambda_2 = -2$. Thus, the solution is

$$y(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

The initial displacement is $y(0) = -1 \text{ m}$, so

$$-1 = C_1 + C_2.$$

Differentiate.

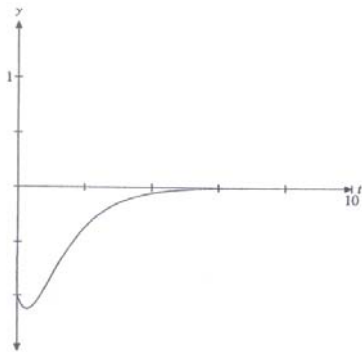
$$y'(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$$

The initial velocity is $y'(0) = -1 \text{ m/s}$, so

$$-1 = -C_1 - 2C_2.$$

Thus, $C_1 = -3$ and $C_2 = 2$, and the solution is

$$y(t) = -3e^{-t} + 2e^{-2t}.$$



9)

The homogeneous equation $y'' + 3y' + 2y = 0$ has characteristic equation $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$ with zeros $\lambda_1 = -1$ and $\lambda_2 = -2$. This leads to the homogeneous solution

$$y_h = C_1 e^{-t} + C_2 e^{-2t}.$$

The particular solution $y_p = Ae^{-4t}$ has derivatives $y_p' = -4Ae^{-4t}$ and $y_p'' = 16Ae^{-4t}$, which when substituted into the equation $y'' + 3y' + 2y = 3e^{-4t}$ provides

$$16Ae^{-4t} + 3(-4Ae^{-4t}) + 2Ae^{-4t} = 3e^{-4t}$$

$$6A = 3$$

$$A = \frac{1}{2}.$$

Thus, a particular solution is $y_p = (1/2)e^{-4t}$. This leads to the general solution

$$y = C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{2} e^{-4t}.$$

The initial condition $y(0) = 1$ provides

$$1 = C_1 + C_2 + \frac{1}{2}.$$

Differentiating,

$$y' = -C_1 e^{-t} - 2C_2 e^{-2t} - 2e^{-4t}.$$

The initial condition $y'(0) = 0$ provides

$$0 = -C_1 - 2C_2 - 2.$$

This system has solution $C_1 = 3$ and $C_2 = -5/2$, leading to the solution

$$y = 3e^{-t} - \frac{5}{2}e^{-2t} + \frac{1}{2}e^{-4t}.$$

10)

A fundamental system of solutions to the homogeneous equation is $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$. We look for a solution of the form

$$y = v_1 y_1 + v_2 y_2 = v_1 \cos 2t + v_2 \sin 2t.$$

Differentiating we get

$$y' = v_1' \cos 2t + v_2' \sin 2t - 2v_1 \sin 2t + 2v_2 \cos 2t.$$

To simplify future calculations we set

$$v_1' \cos 2t + v_2' \sin 2t = 0,$$

$$\text{so } y' = -2v_1 \sin 2t + 2v_2 \cos 2t, \text{ and}$$

$$y'' = -2v_1' \cos 2t + 2v_2' \sin 2t - 4[v_1 \cos 2t + v_2 \sin 2t].$$

Adding, we get

$$y'' + 4y = -2v_1' \cos 2t + 2v_2' \sin 2t = \sec 2t.$$

We must solve the system

$$\begin{aligned} v_1' \cos 2t + v_2' \sin 2t &= 0, \\ -2v_1' \cos 2t + 2v_2' \sin 2t &= \sec 2t. \end{aligned}$$

The solutions are

$$v_1' = -\frac{1}{2} \tan 2t \quad \text{and} \quad v_2' = \frac{1}{2}.$$

Integrating we get

$$v_1(t) = \frac{1}{4} \ln(\cos 2t) \quad \text{and} \quad v_2(t) = \frac{t}{2}.$$

Thus the solution is

$$\begin{aligned} y(t) &= v_1 \cos 2t + v_2 \sin 2t \\ &= \frac{1}{4} \cos 2t \cdot \ln(\cos 2t) + \frac{t}{2} \sin 2t. \end{aligned}$$

11)

We will use the complex method, and look for a solution of the equation $z'' + 5z' + 4z = 2e^{2it}$ of the form $z(t) = ae^{2it}$. Then our particular solution will be $x_p = \text{Im } z$. Differentiating we get

$$\begin{aligned} z'' + 5z' + 4z &= a((2i)^2 + 5(2i) + 4)e^{2it} \\ &= 10aie^{2it} = 2e^{2it}. \end{aligned}$$

Hence $a = -i/5$, $z(t) = -ie^{2it}/5$, and $x_p(t) = \text{Im } z(t) = -\cos 2t/5$.

The characteristic polynomial is $P(\lambda) = \lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4)$, which has roots -1 and -4 . Hence the general solution to the homogeneous equation is $x_h(t) = C_1 e^{-t} + C_2 e^{-4t}$. The general solution to the inhomogeneous equation is

$$x(t) = -\frac{1}{5} \cos 2t + C_1 e^{-t} + C_2 e^{-4t}.$$

The initial conditions imply that

$$\begin{aligned} 1 &= x(0) = -\frac{1}{5} + C_1 + C_2 \\ 0 &= x'(0) = -C_1 - 4C_2 \end{aligned}$$

Hence we find C_1 and C_2 which solve the equations

$$\begin{aligned} C_1 + C_2 &= \frac{6}{5}, \\ -C_1 - 4C_2 &= 0 \end{aligned}$$

The solutions are $C_1 = 8/5$ and $C_2 = -2/5$, so the solution to the initial value problem is

$$x(t) = -\frac{1}{5} \cos 2t + \frac{8}{5} e^{-t} - \frac{2}{5} e^{-4t}.$$

The steady-state response is the particular solution $x_p(t) = -\cos 2t/5$, and the transient response is $x_h(t) = [8e^{-t} - 2e^{-4t}]/5$. In the following plot the graph of the solution to the initial value problem is

the dashed curve, the transient response is dotted, and the steady-state solution is solid.

