

2320  
Test 2 Key

1) The carrying capacity  $K = 20,000$ , and the initial condition  $P_0 = 1000$ , and it is given that  $P(10) = 2000$ . Using equation (1.13), one obtains

$$2000 = \frac{(20000)(1000)}{1000 + (19000)e^{-10r}}$$

Solving gives  $r = -(\ln(9/19))/10 \approx 0.0747$ . After 25 hours, the population is

$$P(25) = \frac{(20000)(1000)}{1000 + (19000)e^{2.5 \ln(9/19)}} \approx 5084.$$

Now, find  $t$  so that  $P(t) = K/2 = 10000$ . This gives

$$e^{\frac{1}{10} \ln(\frac{9}{19})t} = \frac{2000 - 1000}{19000}.$$

That is,  $t = 10(-\ln(19))/(10 \ln(9/19)) \approx 39.4055$ .

2

The model equation,  $I' + I/10 = (1/2)e^{-t/20}$  with  $I(0) = 0$ , has solution  $I = 10(e^{-t/20} - e^{-t/10})$ . The solution is maximized at  $t = 20 \ln 2 \approx 13.86$  with maximum value  $I(20 \ln 2) = 2.5$

3

Let  $y = e^{\lambda t}$  in  $6y'' + 5y' - 6y = 0$  to obtain

$$6\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} - 6e^{\lambda t} = 0,$$

$$e^{\lambda t}(6\lambda^2 + 5\lambda - 6) = 0.$$

Because  $e^{\lambda t} \neq 0$ , we arrive at the logistic equation

$$6\lambda^2 + 5\lambda - 6 = 0$$

$$(3\lambda - 2)(2\lambda + 3) = 0,$$

and roots  $\lambda = 2/3$  and  $\lambda = -3/2$ . Because the roots are distinct, the solutions  $y_1(t) = e^{(2/3)t}$  and  $y_2(t) = e^{-(3/2)t}$  form a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{-(3/2)t} + C_2 e^{(2/3)t}.$$

4

If  $y'' + 2y' + 2y = 0$ , then the characteristic equation is

$$\lambda^2 + 2\lambda + 2 = 0.$$

The roots of the characteristic equation are  $-1 \pm i$ , leading to the complex solutions

$$z(t) = e^{(-1+i)t} \quad \text{and} \quad \bar{z}(t) = e^{(-1-i)t}.$$

However, by Euler's identity,

$$z(t) = e^{-t} e^{it} = e^{-t}(\cos t + i \sin t),$$

and the real and imaginary parts of  $z$  lead to a fundamental set of real solutions,  $y_1(t) = e^{-t} \cos t$  and  $y_2(t) = e^{-t} \sin t$ . Hence, the general solution is

$$y(t) = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t.$$

5

If  $y'' - 4y' + 13y = 0$ , then the characteristic equation is  $\lambda^2 - 4\lambda + 13 = 0$  with roots  $z = 2 \pm 3i$ . The complex solution

$$z(t) = e^{(2+3i)t} = e^{2t}(\cos 3t + i \sin 3t)$$

provides a fundamental set of real solutions,  $y_1(t) = e^{2t} \cos 3t$  and  $y_2(t) = e^{2t} \sin 3t$ , and the general solution

$$y(t) = e^{2t}(C_1 \cos 3t + C_2 \sin 3t).$$

The initial condition  $y(0) = 4$  provides

$$4 = C_1.$$

Differentiating the general solution,

$$y'(t) = 2e^{2t}(C_1 \cos 3t + C_2 \sin 3t)$$

$$+ e^{2t}(-3C_1 \sin 3t + 3C_2 \cos 3t),$$

and the initial condition  $y'(0) = 0$  provides

$$0 = 2C_1 + 3C_2.$$

Thus,  $C_1 = 4$  and  $C_2 = -8/3$ , giving the solution

$$y(t) = e^{2t} \left( 4 \cos 3t - \frac{8}{3} \sin 3t \right).$$

6

By Hooke's Law,

$$k = \frac{F}{y} = \frac{mg}{y}$$

$$k = \frac{(0.05 \text{ kg})(9.8 \text{ m/s}^2)}{(0.2 \text{ m})}$$

$$k = 2.45 \text{ N/m.}$$

Hence,  $my'' + \mu y' + ky = 0$  becomes  $0.05y'' + \mu y' + 2.45y = 0$ , or

$$y'' + 20\mu y' + 49y = 0.$$

This system has characteristic equation  $\lambda^2 + 20\mu\lambda + 49 = 0$ , with zeros given by

$$\lambda = \frac{-20\mu \pm \sqrt{400\mu^2 - 196}}{2}.$$

The system is critically damped if it has one single, repeated root. This happens only if

$$400\mu^2 - 196 = 0$$

$$\mu^2 = \frac{196}{400}$$

$$\mu = \frac{14}{20}.$$

Thus,  $\mu = 7/10$ , and our equation becomes

$$y'' + 14y' + 49y = 0,$$

whose characteristic equation  $\lambda^2 + 14\lambda + 49 = (\lambda + 7)^2 = 0$  has a repeated root  $\lambda = -7$ . Thus, the general solution is

$$y(t) = (C_1 + C_2t)e^{-7t}.$$

If we assume that the mass is displaced in a downward direction,  $y(0) = -0.15 \text{ m}$  and  $C_1 = -0.15$ . Differentiate.

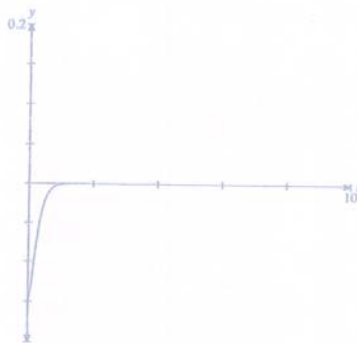
$$y'(t) = C_2e^{-7t} - 7(C_1 + C_2t)e^{-7t}$$

If the mass is released from rest, then  $y'(0) = 0$  and

$$0 = C_2 - 7C_1.$$

Thus,  $C_2 = -1.05$  and the solution is

$$y(t) = -(0.15 + 1.05t)e^{-7t}.$$



7

The homogeneous equation  $y'' + 4y' + 4y = 0$  has characteristic equation  $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$  and repeated root  $\lambda = -2$ . Thus the homogeneous solution is

$$y_h = (C_1 + C_2t)e^{-2t}.$$

The particular solution  $y_p = at + b$  has derivatives  $y_p' = a$  and  $y_p'' = 0$ , which when substituted in  $y'' + 4y' + 4y = 4 - t$ ,

$$4a + 4(at + b) = 4 - t$$

$$4at + (4a + 4b) = -t + 4.$$

Comparing coefficients,

$$4a = -1$$

$$4a + 4b = 4,$$

which has solution  $a = -1/4$  and  $b = 5/4$ . Thus, the general solution is

$$y = (C_1 + C_2t)e^{-2t} - \frac{1}{4}t + \frac{5}{4}.$$

The initial condition  $y(0) = -1$  provides

$$-1 = C_1 + \frac{5}{4}.$$

Differentiate.

$$y' = C_2e^{-2t} - 2e^{-2t}(C_1 + C_2t) - \frac{1}{4}$$

The initial condition  $y'(0) = 0$  provides

$$0 = C_2 - 2C_1 - \frac{1}{4}.$$

Thus,  $C_1 = -9/4$  and  $C_2 = -17/4$  and

$$y = \left(-\frac{9}{4} - \frac{17}{4}t\right)e^{-2t} - \frac{1}{4}t + \frac{5}{4}.$$